FACTORIALITY AND NÉRON-SEVERI GROUPS OF A PROJECTIVE CODIMENSION TWO COMPLETE INTERSECTION WITH ISOLATED SINGULARITIES

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Abstract. For a projective variety Z and for any integer p, define the p-th Néron-Severi group $NS_p(Z)$ of Z as the image of the cycle map $A_p(Z) \to H_{2p}(Z;\mathbb{C})$. Now let $X \subset \mathbb{P}^{2m+1}$ $(m \geq 1)$ be a projective variety of dimension $2m-1$, with isolated singularities, complete intersection of a smooth hypersurface of degree k, with a hypersurface of degree $n > max\{k, 2m+1\}$, and let F be a general hypersurface of degree n containing X. We prove that the natural map $NS_m(X) \to NS_m(F)$ is surjective, and that if $dim\, NS_m(F) = 1$ then $dim\, NS_m(X) = 1$. In particular $dim\, NS_m(X) = 1$ if and only if $dim\, NS_m(F) = 1$. When X is a threefold (i.e. $m = 2$) we deduce a new characterization for the factoriality of X, i.e. that X is factorial if and only if $dim\, NS_2(F) = 1$. This allows us to give examples of factorial threefolds, in some case with many singularities. During the proof of the announced results, we show that the quotient of the middle cohomology of F by the cycle classes coming from X is irreducible under the monodromy action induced by the hypersurfaces of degree n containing X. As consequences we deduce a Noether-Lefschetz Theorem for a projective complete intersection with isolated singularities, and, also using a recent result on codimension two Hodge conjecture, in the case $X \subset \mathbb{P}^5$ is a threefold as before, we deduce that the general hypersurface F of degree n containing X verifies Hodge conjecture.

Keywords and phrases: Projective variety, factoriality, Néron-Severi group, Noether-Lefschetz Theory, Hodge conjecture, monodromy representation, complete intersection isolated singularity, Milnor fibre.

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1. Introduction

Let $X \subset \mathbb{P}^5$ be a complex, projective, complete intersection threefold, with isolated singularities. One says that X is *factorial* if its graded ring is unique factorization domain. This is equivalent to the fact that every surface lying in X is a complete intersection on X ([14], p. 69). Using Lefschetz type Theorems ([1], pg. 50-51), one sees that factoriality is also equivalent to say that every Weil divisor of X is a Cartier divisor, that it is also equivalent to say that X is $\mathbb{Q}$-*factorial*, i.e. that every Weil divisor of X has a multiple which is a Cartier divisor ([1], pg. 5-6),

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and that it is also equivalent to say that X is *locally factorial* ([14], pg. 69). For the interest in this notion in the study of birational geometry, we refer to [17], [23], [2] and [22].

If X is nonsingular then it is factorial. This is no longer true when X is singular. For instance, when X is a complete intersection of general hypersurfaces containing a fixed plane, then $Sing(X)$ consists of $(n+k-2)^2-(n-1)(k-1)$ ordinary double points, where (k,n) is the bi-degree of X, and X is not factorial for it contains the given plane. However it is known that if X is a complete intersection on a *smooth* fourfold $G \subset \mathbb{P}^5$ of degree k with a hypersurface of degree $n \geq k$ (e.g. $X \subset \mathbb{P}^4$), and if X presents few ordinary double points with respect to k and n, then it is factorial (see [2], [3], [4], [5], [6], [7]). Actually one conjectures that previous number is the sharp bound, i.e. that any X as above, with only ordinary double points and such that $|Sing(X)| < (n+k-2)^2-(n-1)(k-1)$, is factorial.

When all the singularities of X are ordinary double points, i.e. are nodes, an important invariant in the study of the birational geometry of X, introduced by C.H. Clemens (see [8], [28]), is the defect $\delta(X)$ of X, which is equal to the difference between the fourth and the second Betti number of X. If X is a complete intersection, the second Betti number is equal to 1 and so we have

$$rk\, H_4(X;\mathbb{Z}) = 1 + \delta(X).$$

If in addition X is a complete intersection on a smooth fourfold $G \subset \mathbb{P}^5$ of degree k with a hypersurface of degree n (e.g. $X \subset \mathbb{P}^4$), then by [9] we know that $\delta(X)$ is the number of dependent conditions that vanishing at singularities of X imposes on the global sections of the line bundle $\mathcal{O}_G(2n+k-6)$ on G. Using the same argument as in [3], Remark 19, one deduces that X is factorial if and only if its defect vanishes, i.e. in the nodal case one has:

$$X \text{ is factorial if and only if } rk\, H_4(X;\mathbb{Z}) = 1, \tag{1}$$

(see [3], [4], [5], [22]). In particular we see that the factoriality can be described by a global topological condition, and that it depends on the position of the nodes in the projective space. Without assuming that the isolated singularities are nodes, characterization (1) holds true only in one direction, i.e.

$$\text{if } rk\, H_4(X;\mathbb{Z}) = 1 \text{ then } X \text{ is factorial.} \tag{2}$$

In fact, if $rk\, H_4(X;\mathbb{Z}) = 1$ then any projective integral surface S contained in X has a general hyperplane section $S \cap H$ which is homologous to a multiple of the hyperplane section of $X \cap H$. Hence by Hamm-Lefschetz Theorem it follows that $S \cap H$ is a complete intersection on $X \cap H$, and so S is on X. As we said before, this means that X is factorial. On the other hand, from Noether-Lefschetz Theorem, the cone $X \subset \mathbb{P}^4$ over a general surface $S \subset \mathbb{P}^3$ of degree $n \geq 4$ is factorial, but $H_4(X;\mathbb{Z}) \simeq H_2(S;\mathbb{Z})$ (see [10], p. 169, (4.18)) and the last group has rank > 1.

Therefore the converse of property (2) is false. Notice that property (2) allows us to give examples of factorial non-nodal threefolds X. In fact, by [10], Theorem (4.17), we know that *if* $X \subset \mathbb{P}^4$ *has just one ordinary singular point of multiplicity* $< deg(X)$*, then* $rk(H_4(X;\mathbb{Z})) = 1$.

Using the notion of Néron-Severi group, one may reformulate property (2) in the following way. First recall that for a projective variety Z and for any integer p, one may define the p-th Néron-Severi group $NS_p(Z)$ of Z as the image of the cycle map $A_p(Z) \to H_{2p}(Z;\mathbb{C})$ (see [12], Chapter 19). Next recall that for a projective complete intersection Z of dimension $2m+\epsilon-1$, $0 \leq \epsilon \leq 1$, with isolated singularities, the only interesting Néron-Severi group is $NS_m(Z)$ (see [10], p. 161, Theorem (4.3)). Now notice that previous argument in proving property (2) works well also if one simply assumes that $dim(NS_2(X)) = 1$, and so, for a threefold X complete intersection with isolated singularities, we have:

$$X \text{ is factorial if and only if } dim(NS_2(X)) = 1. \tag{3}$$

In the present paper we show a new characterization of the factoriality. Roughly saying, we prove that X *is factorial if and only if it is a complete intersection on a smooth fourfold* F *such that* $dim(NS_2(F)) = 1$. More precisely, first we prove the following general result:

Theorem 1.1. *Let* $X \subset \mathbb{P}^{2m+1}$ *be a projective variety of dimension* $2m-1 \geq 1$*, complete intersection of two smooth hypersurfaces* F *and* G *of degrees* n *and* k*, with* $n > k$*. Then* X *has isolated singularities. Moreover, if* $dim\, NS_m(F) = 1$ *then* $dim\, NS_m(X) = 1$.

The property that a complete intersection of two smooth hypersurfaces of different degrees has at most isolated singularities is a direct consequence of Proposition 4.3.6. in [11], and it holds true also when intersecting two smooth hypersurfaces in a projective space of *even* dimension (see also [19], Example 6.3.8.). On the contrary, in the projective space $\mathbb{P}^{2m+2}$ the assertion *if* $dim\, NS_m(F) = 1$ *then* $dim\, NS_m(X) = 1$ is false. In fact, for any smooth hypersurface F of odd dimension, one has $dim\, NS_m(F) = 1$, but there exist smooth complete intersections X of dimension $2m$ with $dim\, NS_m(X) > 1$.

Even if one assumes that X has isolated singularities, the hypothesis G smooth in Theorem 1.1 is necessary, as Example 5 in [3] proves. Next we prove that, conversely, the hypotheses that F is smooth and that $dim\, NS_m(F) = 1$ are also necessary for the property $dim\, NS_m(X) = 1$, in the following sense:

Theorem 1.2. *Let* $X \subset \mathbb{P}^{2m+1}$ *be a projective variety of dimension* $2m-1 \geq 1$*, with isolated singularities, complete intersection of a smooth hypersurface* G *of degree* k*, with a hypersurface of degree* n*. Set* $\mathcal{L} = \mid \mathcal{I}_{X,\mathbb{P}^{2m+1}}(n) \mid$*, and let* $F \in \mathcal{L}$ *be a*

general hypersurface. If $n \geq k$ then F is smooth. Moreover, if $n > max\{k, 2m+1\}$ then the natural map $NS_m(X) \to NS_m(F)$ is surjective.

We will see that the analogous claim of Theorem 1.2 in a projective space of even dimension remains true, but in this case the fact that the map $NS_m(X) \to NS_m(F)$ is surjective is trivial. In fact, as we said, for a smooth hypersurface F of odd dimension, one has $dim\, NS_m(F) = 1$. Previous Theorem 1.1 and Theorem 1.2 give the following Corollary 1.3, from which, taking into account (3), we obtain the announced characterization for the factoriality of threefolds:

Corollary 1.3. *Let $X \subset \mathbb{P}^{2m+1}$ be a projective variety of dimension $2m-1 \geq 1$, with isolated singularities, complete intersection of a smooth hypersurface G of degree k, with a hypersurface of degree n. Set $\mathcal{L} = \mid \mathcal{I}_{X,\mathbb{P}^{2m+1}}(n) \mid$, and let $F \in \mathcal{L}$ be a general hypersurface. If $n \geq k$ then F is smooth. Moreover, if $n > max\{k, 2m+1\}$ then $dim\, NS_m(X) = 1$ if and only if $dim\, NS_m(F) = 1$. In particular, when X is a threefold (i.e. $m = 2$) then X is factorial if and only if $dim\, NS_2(F) = 1$.*

Theorem 1.2 should be compared with [21], where one describes the Picard group of a general complete intersection surface containing a fixed smooth curve. Observe also that, assuming $k = 1$, all previous results apply to any hypersurface $X \subset \mathbb{P}^{2m}$ of degree n, with isolated singularities.

The line of the proof of Theorem 1.1 is the following. As we said, X only has isolated singularities by Proposition 4.3.6. in [11]. Next, in order to prove that any projective subvariety $S \subset X$ of dimension m is homologous to a multiple of the linear section H_X^{m-1} of X, using [12], Example 15.3.2, we may assume that S is integral with isolated singularities. For such a subvariety, using our hypotheses on $NS_m(F)$, the positivity assumption $n > k$ and a suitable application of Hodge Index Theorem for G, we are able to compare the double point formulae relative to the inclusions $S \subset F$, $S \subset G$ and $S \cap R \subset X \cap R$, where R is a general hypersurface of any degree $r \geq 1$ (so that $S \cap R$ and $X \cap R$ are smooth). It turns out that $S \cap R$ is homologous to a multiple $lH_{X\cap R}^{m-1}$ of the linear section $H_{X\cap R}^{m-1}$ of $X \cap R$ in $H^{2m-2}(X \cap R; \mathbb{C})$. To lift this homology to the whole X, we consider a general hypersurface R of degree $r \geq 1$, and a general pencil $\rho : \tilde{X} \to \mathbb{P}^1$ of hypersurface sections $\rho^{-1}(t) = X \cap R_t$ of X of degree r with $R = R_0$ ($\tilde{X}$ = blowing-up of X along the exceptional subset of the pencil). The image $\tau(S - lH_X^{m-1})$ of the cycle $S - lH_X^{m-1}$ through the Gysin morphism $\tau : H_{2m}(X; \mathbb{C}) \to H_{2m}(\tilde{X}; \mathbb{C})$, maps to 0 in $H_{2m}(\tilde{X}, \rho^{-1}(\mathcal{C}); \mathbb{C})$ ($\mathcal{C}$ = critical locus of the pencil) because, using the Invariant Subspace Theorem (see [25], p.165-166), one may prove that $H_{2m}(\tilde{X}, \rho^{-1}(\mathcal{C}); \mathbb{C})$ is canonically embedded in $H^{2m-2}(X \cap R_t; \mathbb{C})$ (here we need that the dimension of X is odd). Therefore $\tau(S - lH_X^{m-1})$ comes from $H_{2m}(\rho^{-1}(\mathcal{C}); \mathbb{C})$, and so it is 0, because by [10], Theorem (4.3), p. 161, we know that this space is generated by

the linear sections of the singular fibres parametrized by $\mathcal{C}$. It follows that also $S - lH_X^{m-1}$ is 0 (i.e. $S = lH_X^{m-1}$ in $H_{2m}(X;\mathbb{C})$) because τ is injective.

More generally, the same argument we previously used to lift the homology of algebraic cycles to X applies to any cycle, i.e. one has the following:

Proposition 1.4. *Let $X \subset \mathbb{P}^N$ be a complete intersection projective variety of odd dimension $2m-1 \geq 1$, with isolated singularities. Then for a general hypersurface R of degree $r \geq 1$ the Gysin morphism $H_{2m}(X;\mathbb{C}) \to H_{2m-2}(X \cap R;\mathbb{C})$ is injective.*

Observe that when X is smooth, then previous Proposition 1.4 follows from Lefschetz Hyperplane Theorem. We need Proposition 1.4 for a comment on Theorem 1.8 below.

As for the proof of Theorem 1.2, first we prove that F is smooth using a Bertini type of argument (which holds true also when X is of codimension two in $\mathbb{P}^{2m+2}$). Next we show that, in a certain sense, the classical Noether-Lefschetz argument applies in our setting. More precisely, denote by $\mathcal{Q} \subset \mathbb{P}^N$ the image of $\mathbb{P}^{2m+1}$ through the rational map $\mathbb{P}^{2m+1} - - \to \mathbb{P}^N$ defined choosing a basis of the linear system $\mathcal{L}$ ($N = dim\mathcal{L}$). The variety $\mathcal{Q}$ only has isolated singularities, and so one may regard F as a general hyperplane section of $\mathcal{Q}$, and may vary it in a general pencil $L \subset \mathbb{P}^{N^*}$ of hyperplane sections. The monodromy action of the pencil induces an orthogonal decomposition

$$H^{2m}(F;\mathbb{C}) = I \oplus V, \tag{4}$$

where I is the subspace of the invariant cocycles, and V is its orthogonal complement. Using a standard argument, we reduce the proof of Theorem 1.2 to prove that the monodromy of the pencil irreducibly acts on V, i.e. we prove the following:

Theorem 1.5. *The monodromy representation on V for the family of hypersurfaces of degree n containing X is irreducible.*

To prove this (see also Remark 3.3 below), using [18] and the theory of isolated singular points on complete intersections as developed in [20], first we prove that V is generated by the vanishing cocycles corresponding to the hyperplane sections of $\mathcal{Q}$ which are *tangent* at some regular point of $\mathcal{Q}$, and by a certain subspace of the space generated by the vanishing cocycles defined by the *remaining* singular hyperplane sections, i.e. by the hyperplane sections of $\mathcal{Q}$ passing through its singularities (except for the singularity of $\mathcal{Q}$ coming from the contraction of G). Next we prove a basic lemma (i.e. Lemma 3.2 below) which states that the monodromy *trivially* acts on the vanishing cocycles of the latter type. This implies that V only is generated by the vanishing cocycles coming from tangential sections, and so we may conclude the proof of Theorem 1.5 using the classical Zariski Theorem. To prove Lemma 3.2,

using [20], first we reduce it to the case X is a complete intersection like $G \cap F'$, with G a general hypersurface and F' a general hypersurface with a unique double point q_1 also belonging to G. Then we conclude the proof of Lemma 3.2 by an "ad hoc" argument, relying on the fact that one may realize the Milnor fibre of a general element F'' of the linear system $\mathcal{L}$ passing through q_1, as contained in a *fixed* sphere which does not depend on F'' (see (35) below). This argument does not apply to those F'' corresponding to limit of tangential hyperplane sections of $\mathcal{Q}$ for which there exists a sequence of regular contact points converging to q_1. This case requires a separate analysis, which in turn relies on the fact that, being F' general, such F'' are parametrized by the dual variety of the tangent cone of $\mathcal{Q}$ at q_1, which is a nondegenerate and irreducible quadric in the projective space parametrizing the hypersurfaces passing through q_1.

Combining Theorem 1.5 with [16], Corollary 1.1, we obtain the following corollary:

Corollary 1.6. *Let $X \subset \mathbb{P}^5$ be a projective threefold with isolated singularities, complete intersection of a smooth hypersurface of degree k, with a hypersurface of degree n. Assume $n > max\{k, 5\}$, set $\mathcal{L} = \mid \mathcal{I}_{X,\mathbb{P}^5}(n) \mid$, and let $F \in \mathcal{L}$ be a general hypersurface. Then Hodge conjecture holds true for F.*

In fact, Theorem 1.5 implies that $H^{2,2}(F) \subset I$. Hence all the Hodge cycles of F come from the Hodge cycles of a desingularization of $\mathcal{Q}$, i.e. from some rational projective complex manifold of dimension 5, for which, by [16], Corollary 1.1 (here we are forced to assume $X \subset \mathbb{P}^5$), Hodge conjecture holds true.

Both Theorem 1.5 and Corollary 1.6 should be compared with ([24], Conjecture 0.2, Theorems 0.3 and 0.4, and Corollary 0.5), where the authors prove similar results, with different assumptions. To this purpose, let we make two remarks.

First we notice one may prove that the subspace $I \subset H^{2m}(F;\mathbb{C})$ defined by decomposition (4) is the image of $H_{2m}(X;\mathbb{C})$ in $H_{2m}(F;\mathbb{C}) \simeq H^{2m}(F;\mathbb{C})$, and so, similarly as in [24], the subspace V (for which our Theorem 1.5 states the irreducibility) is nothing but the quotient of $H^{2m}(F;\mathbb{C})$ by the cycle classes coming from X. In other words, with an analogous notation as in [24], one has $V = H^{2m}(F;\mathbb{C})^{\mathrm{van}}_{\perp X}$, and we may restate previous Theorem 1.5 as follows:

Theorem 1.7. *Let $X \subset \mathbb{P}^{2m+1}$ be a projective variety of dimension $2m-1 \geq 1$, with isolated singularities, complete intersection of a smooth hypersurface of degree k, with a hypersurface of degree $n > k$. Then the monodromy representation on $H^{2m}(F;\mathbb{C})^{van}_{\perp X}$ for the family of hypersurfaces F of degree n containing X is irreducible.*

For the proof see *Proof of Theorem 1.7* in Section 3 below.

Next consider a projective surface $Z \subset \mathbb{P}^5$ whose ideal is generated in degrees $\geq \delta$. Under mild assumptions on the singularities of Z, one knows that Z is contained in smooth hypersurfaces G and F of degree $k = \delta + 1$ and $n = \delta + 2$ (see [24]). From our Theorem 1.1 we also know that $X = G \cap F$ only has isolated singularities, and therefore from Corollary 1.6 (when $\delta > 3$) the general hypersurface F of degree n containing X verifies Hodge conjecture. A fortiori this holds true for a general hypersurface of degree $n = \delta + 2$ containing Z. So we see that, in the case of a family of hypersurfaces of $\mathbb{P}^5$, our Corollary 1.6 (at least when $\delta > 3$, and for that concerns the assertion on Hodge conjecture) implies Corollary 0.5 in [24].

As a further consequence of the proof of our Theorem 1.5, we may state a Noether-Lefschetz type Theorem for complete intersections $\mathcal{Q}$ with isolated singularities, i.e. we are able to prove the following:

Theorem 1.8. *(Noether-Lefschetz Theorem with isolated singularities) Let $\mathcal{Q} \subset \mathbb{P}^N$ be an irreducible complete intersection projective variety with isolated singularities, of odd dimension $2m+1 \geq 3$. Assume that $dim(NS_{m+1}(\mathcal{Q})) = 1$. Then for any integer $r >> 0$ and any general hypersurface R of degree r one has $dim(NS_m(\mathcal{Q} \cap R)) = 1$.*

For the proof see *Proof of Theorem 1.8* in Section 3 below. Observe that, in view of Proposition 1.4, the assumption $dim(NS_{m+1}(\mathcal{Q})) = 1$ in Theorem 1.8 is necessary.

Finally we point out that, using Theorem 1.1, we are able to construct complete intersections X of dimension $2m-1 \geq 3$ with isolated singularities and with $dim(NS_m(X)) = 1$ (e.g. factorial threefolds with isolated singularities), in some cases also with many singularities: see Corollary 2.2 and Corollary 2.3 below. In particular we prove that the asymptotic behavior of the maximal integer r for which there exists a nodal factorial threefold in $\mathbb{P}^5$ complete intersection of a smooth hypersurface of degree $n-1$ with a hypersurface of degree n, with $|Sing(X)| = r$, is n^5. We also stress that from [10], Theorem (4.5), one may deduce that for a nodal hypersurface $X \subset \mathbb{P}^{2m}$ of degree n with at most $m(n-2)$ nodes, one has $dim(NS_m(X)) = 1$.

Now we are going to prove the announced results.

2. Proof of Theorem 1.1 and consequences

Proof of Theorem 1.1. By Proposition 4.3.6. in [11] it follows that X has at most isolated singularities.

Now fix an integral subvariety $S \subset X$ of dimension m. To prove Theorem 1.1 it suffices to prove that S is homologous in X to a multiple of the m-dimensional

linear section of X. To this aim first notice that, by [12], Example 15.3.2, we may assume $Sing(S) \subset Sing(X)$, i.e. we may assume S with isolated singularities. In particular, if $R \subset \mathbb{P}^{2m+1}$ denotes a general hypersurface of degree $r \geq 1$, then $C = S \cap R$ and $Y = X \cap R$ are smooth projective varieties of dimensions $m-1$ and $2m-2$, with $C \subset Y$. In $H^{2m-2}(Y;\mathbb{C})$ we may write

$$C = \frac{d}{kn} H_Y^{m-1} + \alpha,$$

where d is the degree of S, H_Y is the general hyperplane section of Y, and $\alpha \in H^{2m-2}(Y;\mathbb{C})$ is a primitive class, i.e. $\alpha.H_Y = 0$ in $H^{2m}(Y;\mathbb{C})$. We deduce:

$$(C.C)_Y = \frac{d^2 r}{kn} + \alpha^2,$$

where $(C.C)_Y$ denotes the self-intersection of C in Y. On the other hand, from the double point formula ([12], p. 166), we know that

$$(C.C)_Y = (c(i^* T_Y) c(T_C)^{-1})_{m-1},$$

where i denotes the inclusion $C \subset Y$, T_Y and T_C the tangent bundles, and c the total Chern class. Putting together we obtain:

$$\alpha^2 = (c(i^* T_Y) c(T_C)^{-1})_{m-1} - \frac{d^2 r}{kn}. \tag{5}$$

Besides previous double point formula we may also consider the two double point formulae corresponding to the inclusions $S \subset F$ and $S \subset G$. More precisely, let $\pi : \Sigma \to S$ be a desingularization of S. Denote by $f : \Sigma \to F$ and $g : \Sigma \to G$ the compositions of π with the natural inclusions. Following [12], p. 166, denote by $\widetilde{\Sigma \times \Sigma}$ the blowing-up along the diagonal, by $\tilde{D}(f) \subset \widetilde{\Sigma \times \Sigma}$ and $\tilde{D}(g) \subset \widetilde{\Sigma \times \Sigma}$ the double point schemes of f and g, by $D(f) \subset \Sigma$ and $D(g) \subset \Sigma$ the double point sets, and by $\mathbb{D}(f) \in A_0(D(f))$ and $\mathbb{D}(g) \in A_0(D(g))$ the double point classes. Applying the double point formula to f and g we obtain

$$deg(\mathbb{D}(f)) = (S.S)_F - (c(f^* T_F) c(T_\Sigma)^{-1})_m \tag{6}$$

and

$$deg(\mathbb{D}(g)) = (S.S)_G - (c(g^* T_G) c(T_\Sigma)^{-1})_m, \tag{7}$$

where $(S.S)_F$ and $(S.S)_G$ represent the self-intersection of S in F and in G.

We claim that

$$deg(\mathbb{D}(f)) = deg(\mathbb{D}(g)). \tag{8}$$

To prove this, denote by φ and γ the natural maps $\widetilde{\Sigma \times \Sigma} \to F \times F$ and $\widetilde{\Sigma \times \Sigma} \to G \times G$, and by $\Delta_F \subset F \times F$ and $\Delta_G \subset G \times G$ the diagonals. Recall that $\tilde{D}(f)$ and $\tilde{D}(g)$ are defined as the residual schemes to the exceptional divisor E of $\widetilde{\Sigma \times \Sigma}$, in $\varphi^{-1}(\Delta_F)$ and in $\gamma^{-1}(\Delta_G)$. Since $\varphi^{-1}(\Delta_F) = \gamma^{-1}(\Delta_G)$, then we have

$$\tilde{D}(f) = \tilde{D}(g). \tag{9}$$

From (9) and ([12], p. 166), it follows that to prove (8) it suffices to show that the residual intersection classes $\tilde{\mathbb{D}}(f)$ and $\tilde{\mathbb{D}}(g)$ coincide in $A_0(\tilde{D}(f)) = A_0(\tilde{D}(g))$. To this purpose, notice that by ([12], Theorem 9.2) we have

$$\tilde{\mathbb{D}}(f) = \{c(N_{\Delta_F} \otimes \mathcal{O}(-E)) \cap s(\tilde{D}(f), \widetilde{\Sigma \times \Sigma})\}_0 \tag{10}$$

and

$$\tilde{\mathbb{D}}(g) = \{c(N_{\Delta_G} \otimes \mathcal{O}(-E)) \cap s(\tilde{D}(g), \widetilde{\Sigma \times \Sigma})\}_0, \tag{11}$$

where c and s denote Chern and Segre classes, and $\mathcal{O}(-E)$, N_{Δ_F} and N_{Δ_G} are the pull-back on $\varphi^{-1}(\Delta_F) = \gamma^{-1}(\Delta_G)$ of $\mathcal{O}_{\widetilde{\Sigma\times\Sigma}}(-E)$ and of the normal bundles of Δ_F and Δ_G in $F \times F$ and $G \times G$. Since these normal bundles are isomorphic to the tangent bundles of F and G and the Chern polynomials of both F and G only depend on the hyperplane class, and since π is an isomorphism outside of a finite set of S, then both $c(N_{\Delta_F})$ and $c(N_{\Delta_G})$ are the identity in $A(\tilde{D}(f)) = A(\tilde{D}(g))$. In particular $c(N_{\Delta_F}) = c(N_{\Delta_G})$ in $A(\tilde{D}(f)) = A(\tilde{D}(g))$. Therefore from (9), (10) and (11), we obtain (8).

Now we notice that our assumption on $NS_m(F)$ implies that $(S.S)_F = \frac{d^2}{n}$. On the other hand, by Hodge Index Theorem for G (see [15], Theorem 5.2, pg. 435), we have $(-1)^m(S - \frac{d}{k}H_G^m)^2 \geq 0$ on G (H_G = general hyperplane section of G), and so $(-1)^m((S.S)_G - \frac{d^2}{k}) \geq 0$. Comparing with (5), (6), (7) and (8), and taking into account that $k < n$, we obtain that $(-1)^{m-1}\alpha^2$ is less than or equal to

$$(-1)^{m-1}\left[(c(i^*T_Y)c(T_C)^{-1})_{m-1} - \frac{r}{k-n}\left[(c(f^*T_F) - c(g^*T_G))c(T_\Sigma)^{-1}\right]_m\right]. \tag{12}$$

Using the exact sequences $0 \to T_C \to T_\Sigma|_C \to \mathcal{O}_C(r) \to 0$, $0 \to T_Y \to T_F|_Y \to \mathcal{O}_Y(r) \oplus \mathcal{O}_Y(k) \to 0$ and $0 \to T_Y \to T_G|_Y \to \mathcal{O}_Y(r) \oplus \mathcal{O}_Y(n) \to 0$, one may compare the intersection numbers appearing in (12). It turns out that the number in the formula (12) is 0, therefore we have:

$$(-1)^{m-1}\alpha^2 \leq 0.$$

By Hodge Index Theorem for Y it follows that $\alpha = 0$. In other words, for any integer $r \geq 1$ and for a general hypersurface $R \subset \mathbb{P}^{2m+1}$ of degree r, one has

$$S \cap R = \frac{d}{kn} H_{X\cap R}^{m-1} \quad \text{in} \quad H^{2m-2}(X \cap R; \mathbb{C})$$

($H_{X\cap R}$ = general hyperplane section of $X \cap R$). At this point, to conclude the proof of Theorem 1.1 it suffices to prove Proposition 1.4.

Proof of Proposition 1.4. Consider a general pencil $\{X \cap R_t\}_{t\in\mathbb{P}^1}$ of hypersurface sections of X, with $deg(R_t) = r$. We may regard the pencil as the set of fibres of a projective morphism

$$\rho : \tilde{X} \to \mathbb{P}^1,$$

where $\tilde{X}$ is the blowing-up of X along the exceptional subset. Let $\mathcal{C} \subset \mathbb{P}^1$ be the critical locus of ρ, put $U = \mathbb{P}^1 - \mathcal{C}$, and consider the natural exact sequence

$$H_{2m}(\rho^{-1}(\mathcal{C});\mathbb{C}) \to H_{2m}(\tilde{X};\mathbb{C}) \to H_{2m}(\tilde{X},\rho^{-1}(\mathcal{C});\mathbb{C}). \tag{13}$$

Applying Lefschetz Duality to the pair $(\tilde{X},\rho^{-1}(\mathcal{C}))$ ([26], p. 297), we obtain a natural isomorphism

$$H_{2m}(\tilde{X},\rho^{-1}(\mathcal{C});\mathbb{C}) \simeq H^{2m-2}(\rho^{-1}(U);\mathbb{C}). \tag{14}$$

The Leray spectral sequence of the restriction $\rho^{-1}(U) \to U$ collapses in the term E_2 (see [25], p. 166), and therefore we have an isomorphism:

$$H^{2m-2}(\rho^{-1}(U);\mathbb{C}) \simeq \oplus_{j\geq 0} H^j(U, R^{2m-2-j}\rho_*\mathbb{C}). \tag{15}$$

Since for any $t \in U$ the fiber $\rho^{-1}(t) = X \cap R_t$ is a smooth projective complete intersection of *even* complex dimension $2m-2$, it follows that $H^j(U, R^{2m-2-j}\rho_*\mathbb{C}) = 0$ for any $j \geq 1$ odd. For the same reason, when $j \geq 1$ is even, the local system $R^{2m-2-j}\rho_*\mathbb{C}$ on U has rank 1 and has a global section with no zeros, corresponding to the linear section of X with a general subspace of $\mathbb{P}^{2m+1}$ of codimension $(2m-2-j)/2$. Hence $R^{2m-2-j}\rho_*\mathbb{C}$ is isomorphic to the constant sheaf $\mathbb{C}$. So we have $H^j(U, R^{2m-2-j}\rho_*\mathbb{C}) = H^j(U;\mathbb{C})$, which again vanishes when $j \geq 1$ is even, by Lefschetz Duality (we may assume that $\mathcal{C}$ is non empty). Therefore, from (15), we deduce that the natural map

$$H^{2m-2}(\rho^{-1}(U);\mathbb{C}) \to H^0(U, R^{2m-2}\rho_*\mathbb{C})$$

is an isomorphism. Taking into account that, for any $t \in U$, $H^0(U, R^{2m-2}\rho_*\mathbb{C})$ identifies with the invariant subspace $H^{2m-2}(\rho^{-1}(t);\mathbb{C})^{\mathrm{inv}}$, it follows a natural inclusion:

$$H^{2m-2}(\rho^{-1}(U);\mathbb{C}) \subset H^{2m-2}(\rho^{-1}(t);\mathbb{C}). \tag{16}$$

Now fix a cycle $a \in H_{2m}(X;\mathbb{C})$ which restricts to 0 in $H_{2m-2}(X \cap R_t;\mathbb{C}) \simeq H^{2m-2}(\rho^{-1}(t);\mathbb{C})$ $(t \in U)$, and let $\tau(a)$ be the image of a through the Gysin morphism $\tau : H_{2m}(X;\mathbb{C}) \to H_{2m}(\tilde{X};\mathbb{C})$ (see [12], Example 19.2.1). Using (13), (14) and (16) we see that the map $H_{2m}(\tilde{X};\mathbb{C}) \to H_{2m}(\tilde{X},\rho^{-1}(\mathcal{C});\mathbb{C})$ sends $\tau(a)$ to 0. From the exact sequence (13) it follows that $\tau(a)$ comes from $H_{2m}(\rho^{-1}(\mathcal{C});\mathbb{C})$. This space is the direct sum of a finite number of spaces like $H_{2m}(X \cap R_{t'};\mathbb{C})$, where $X \cap R_{t'}$ is a projective complete intersection of dimension $2m-2$ with isolated singularities. From [10], Theorem (4.3), p. 161, we know that such a space has dimension 1. Hence $\tau(a)$ is equal to a certain multiple of $\tau(H_X^{m-1})$ in $H_{2m}(\tilde{X};\mathbb{C})$. Since a restricts to 0 in $H_{2m-2}(X \cap R_t;\mathbb{C})$ it follows that $\tau(a) = 0$, which implies that $a = 0$ because τ is injective.

This concludes the proof of Proposition 1.4 and, as we said before, the proof of Theorem 1.1. $\square$

One may consider Theorem 1.1 as a method to construct complete intersections X of dimension $2m-1 \geq 3$, with isolated singularities and with $dim(NS_m(X)) = 1$ (e.g. factorial threefolds with isolated singularities). A question arising naturally from this remark is how many (and what kind of) singularities one may produce in this fashion. To this purpose we are able to prove the following Corollary 2.2 and Corollary 2.3. First we need the following:

Lemma 2.1. *Let $F \subset \mathbb{P}^b$ ($b \geq 2$) be a projective smooth hypersurface. Fix an integer $k > 0$, and r points $\Sigma = \{p_1, \dots, p_r\}$ on F. Assume that $r < \left(\frac{b+1+k}{b+3}\right)^b$, and that the ideal of Σ is generated in degree $\leq \frac{b+1+k}{b+3}$. Then there exists a projective smooth hypersurface $G \subset \mathbb{P}^b$ of degree k such that the singular locus of the complete intersection $X = G \cap F$ is Σ, and each point p_i is an ordinary double point for X.*

Proof. Denote by $\pi : \mathbb{P}_1 \to \mathbb{P}^b$ the blowing-up of $\mathbb{P}^b$ along Σ, by $E = \sum_{i=1}^r E_i$ its exceptional divisor, and by $\tilde{H}_i$ the strict transform in $\mathbb{P}_1$ of the tangent hyperplane of F at p_i. Put $H_i = \tilde{H}_i \cap E_i$, denote by $\rho : \mathbb{P}_2 \to \mathbb{P}_1$ the blowing-up of $\mathbb{P}_1$ along $\sum_{i=1}^r H_i$, and by $L = \sum_{i=1}^r L_i$ its exceptional divisor. Consider the linear system:

$$\mid D \mid = \mid \rho^*(k\pi^*(H) - E) - L \mid,$$

where $H \subset \mathbb{P}^b$ denotes a hyperplane divisor. As a first step we prove that $\mid D \mid$ is base-point free.

To this purpose first we prove that $\mid D \mid$ is base-point free on L. To this aim notice that $H_i \simeq \mathbb{P}^{b-2}$, and that its normal bundle in $\mathbb{P}_1$ is isomorphic to $\mathcal{O}_{H_i}(-1) \oplus \mathcal{O}_{H_i}(1)$. Therefore the $\mathbb{P}^1$-bundle $L_i \to H_i$ is isomorphic to $\mathbb{P}(\mathcal{O}_{H_i} \oplus \mathcal{O}_{H_i}(2))$. The divisor D restricts on each component L_i to a moving section of $L_i \to H_i$, which defines a base-point free linear system on L_i. Hence, by the defining sequence of $\mathcal{O}_L(D)$, it follows that to prove $\mid D \mid$ is base-point free on L it suffices to prove $h^1(\mathbb{P}_2, \mathcal{O}_{\mathbb{P}_2}(D-L)) = 0$. On the other hand, since $D - L = K_{\mathbb{P}_2} + (b+1+k)(\pi \circ \rho)^*(H) - b\rho^*(E) - 3L$ ($K_{\mathbb{P}_2}$= canonical divisor of $\mathbb{P}_2$), then by Kawamata-Vieweg Theorem it suffices to prove that the divisor $A = (b+1+k)(\pi\circ\rho)^*(H) - b\rho^*(E) - 3L$ is big and nef.

Since the restrictions of A to the strict transform $\tilde{E}$ of E and to L are very ample, to prove that A is nef it suffices to prove that $A.C_2 \geq 0$ for the strict transform $C_2 \subset \mathbb{P}_2$ of any irreducible curve $C \subset \mathbb{P}^b$. Taking into account that $\rho^*(E).C_2 \geq L.C_2$, and using the projection formula, we see that $A.C_2 \geq ((b+1+k)\pi^*(H) - (b+3)E).C_1$ (C_1= strict transform of C in $\mathbb{P}_1$), which is ≥ 0 for our assumption on the degree of the generators of the ideal of Σ. This proves that A is nef.

To prove that A is big, first we notice that, using previous description of the bundle $L_i \to H_i$ and the intersection formulae for a blowing-up appearing in [12], p. 67, one may prove that ${L_i}^j.\rho^*(E_i)^{b-j} = 1$ for $0 \leq j \leq b$ even, and 0 otherwise. It follows that $A^b \geq \left[(b+1+k)^b - (b+3)^b\right] r$, which is > 0 for our assumption on r. This proves that A is big, and so that $\mid D \mid$ is base-point free on L.

A similar computation proves that $\mid D \mid$ is base-point free on $\tilde{E}$, and out of $\tilde{E}+L$. This proves that $\mid D \mid$ is base-point free. Therefore, by Bertini Theorem, any general divisor $D \in \mid D \mid$ is smooth. Then one may choose $G = (\pi \circ \rho)_*(D)$, because similar argument and computation as before prove that G is isomorphic to D, that $Sing(G \cap F) = \Sigma$, and that each p_i is a node for $G \cap F$. $\square$

We are in position to prove the announced corollaries:

Corollary 2.2. *Fix integers $m \geq 2$, k and r, put $q = \frac{2m+2+k}{2m+4}$, and assume that $r \leq \binom{q}{2m+1}$. Then for any $n > k$ there exists an integral projective complete intersection $X \subset \mathbb{P}^{2m+1}$ of bi-degree (k, n) whose singular locus consists of exactly r ordinary double points, and such that $dim(NS_m(X)) = 1$.*

Proof. Fix r general points $p_1, \dots, p_r$ in $\mathbb{P}^{2m+1}$. Since $r < \binom{n+2m+1}{2m+1}$ then there exists a general hypersurface F of degree n passing through $p_1, \dots, p_r$. By Noether-Lefschetz Theorem we have $dim(NS_m(F)) = 1$. On the other hand, combining our assumption $r \leq \binom{q}{2m+1}$ with Lemma 2.1 and [13], Corollary 1.6, we deduce the existence of a smooth hypersurface G of degree k such that the singular locus of the complete intersection $X = G \cap F$ consists of exactly r nodes at the points $p_1, \dots, p_r$. By our Theorem 1.1, one has $dim(NS_m(X)) = 1$. $\square$

Next result states that the examples exhibited by previous Corollary 2.2 in $\mathbb{P}^5$ are asymptotically sharp, i.e. one has:

Corollary 2.3. *For any integer $n \geq 3$ denote by $\nu(n)$ the maximal integer r for which there exists a nodal factorial threefold in $\mathbb{P}^5$ complete intersection of a smooth hypersurface of degree $n-1$ with a hypersurface of degree n, with $|Sing(X)| = r$. Then there exist positive constants $\gamma_1 > 0$ and $\gamma_2 > 0$ such that*

$$\gamma_1 \leq \frac{\nu(n)}{n^5} \leq \gamma_2$$

for any $n \geq 3$.

Proof. From Corollary 2.2 we deduce that $\frac{\nu(n)}{n^5}$ is bounded from below by some positive constant for $n >> 0$. On the other hand, by [3] and [7] we know that $\nu(n) \geq 1$ for any $n \geq 3$. This proves the existence of γ_1. As for γ_2, recall that the defect of a nodal factorial threefold X vanishes. If such a threefold X is a complete intersection in $\mathbb{P}^5$ of a smooth hypersurface of degree $n-1$ with a hypersurface of degree n, this means that the value of the Hilbert function of the singular locus of X at level $3n-7$ is $|Sing(X)|$ (see [9]). Therefore one has $|Sing(X)| \leq \binom{3n-2}{5}$. $\square$

Our numerical assumption in the proof of Lemma 2.1 certainly is not the best possible. It only is of the simplest form we were able to conceive. We also notice that the proof of Lemma 2.1 can be generalized to worse singularities, and that one

may state a similar result as in Corollary 2.3 for threefolds complete intersections in $\mathbb{P}^5$ of bi-degree (k,n) $(k<n)$ with k not too far from $n-1$. We decided not to push here this investigation further. We have in mind to give more information on this subject in a forthcoming paper.

3. Proof of Theorem 1.2 and consequences

Proof of Theorem 1.2. For any $p \in Sing(X)$, let $\mathcal{L}_p \subset \mathcal{L}$ be the closed set of all $F \in \mathcal{L}$ such that $p \in Sing(F)$. By Bertini Theorem, the singular locus of any general $F \in \mathcal{L}$ is contained in X. Since X is the complete intersection of F and G, then any singular point of F has to be also a singular point for X. In other words, for a general $F \in \mathcal{L}$ one has $Sing(F) \subset Sing(X)$. Therefore, in order to prove that the general $F \in \mathcal{L}$ is smooth, it suffices to prove that, for any $p \in Sing(X)$, $\mathcal{L} - \mathcal{L}_p$ is non empty. To this purpose, fix $D \in \mathcal{L}_p$, and denote by R a hypersurface of degree $n-k$ such that $p \notin R$. Let $D_1 \in \mathcal{L}$ be the hypersurface defined by the equation $d + rg = 0$, where $g = 0$, $d = 0$ and $r = 0$ are the equations defining G, D and R (when $n = k$, define D_1 by the equation $d + g = 0$). Taking into account that G is smooth, computing derivatives one sees that D_1 is smooth at p, and so $\mathcal{L} - \mathcal{L}_p$ is a non empty subset of $\mathcal{L}$.

Next, we are going to prove that if $n > max\{k, 2m+1\}$ then, for a general $F \in \mathcal{L}$, the map $NS_m(X) \to NS_m(F)$ is surjective. The proof is an adaptation of the classical Noether-Lefschetz argument.

Choosing a basis of the linear system $\mathcal{L}$, we may define a rational map:

$$\mathbb{P}^{2m+1} \dashrightarrow \mathbb{P} := \mathbb{P}^N \tag{17}$$

$(N = dim\mathcal{L})$, whose resolution is represented by the blowing-up $\mathcal{P}$ of $\mathbb{P}^{2m+1}$ along X, equipped with natural maps $\mathcal{P} \to \mathbb{P}^{2m+1}$ and $\mathcal{P} \to \mathbb{P}$ (see [15], p. 168). By [12], p. 437, B.6.10., we know that locally $\mathcal{P}$ is the hypersurface of $\mathbb{A}^{2m+1} \times \mathbb{P}^1$ defined by the equation

$$gu_0 - fu_1 = 0, \tag{18}$$

where $g = 0$ and $f = 0$ are the local equations of G and F in $\mathbb{A}^{2m+1}$, and u_0, u_1 are coordinates in $\mathbb{P}^1$. Denote by $\mathcal{Q} \subset \mathbb{P}$ the image of $\mathcal{P}$. The map $\mathcal{P} \to \mathbb{P}$ sends all the points of the strict transform $\tilde{G}$ of G (and only them) to a singular point q_∞ of $\mathcal{Q}$, and, since $n > k$, the global sections of the pull-back of $\mathcal{O}_{\mathbb{P}}(1)$ separate points and tangent vectors of $\mathcal{P}$ out of $\tilde{G}$, i.e. the map $\mathcal{P} \to \mathbb{P}$ induces an algebraic isomorphism

$$\mathcal{P} - \tilde{G} \simeq \mathcal{Q} - \{q_\infty\}. \tag{19}$$

From (18) and (19) one deduces that, besides the point q_∞, the singular locus of $\mathcal{Q}$ consists of a certain finite number of points $q_1, \dots, q_r$ corresponding to the

singular points of X. Hence we have $Sing(\mathcal{Q}) = \{q_1, \dots, q_r, q_\infty\}$, and $Sing(\mathcal{P}) = \{q_1, \dots, q_r\}$ (actually each point $q_i \neq q_\infty$ is a double point, and it is a node if and only if the corresponding singular point of X is).

For any x in the dual space $\mathbb{P}^*$, denote by $H_x \subset \mathbb{P}$ the corresponding hyperplane, and by F_x the corresponding hypersurface of $\mathcal{L}$. Next denote by $\mathcal{D} \subset \mathbb{P}^*$ the discriminant variety of $\mathcal{L}$, i.e. the variety parametrizing the singular hypersurfaces of $\mathcal{L}$. Such a variety has $r+2$ components: the dual variety $\mathcal{Q}^*$ of $\mathcal{Q}$, the hyperplane $\mathcal{H}_\infty$ corresponding to the singular point q_∞ (i.e. corresponding to the reducible hypersurfaces of $\mathcal{L}$ containing G), and the r hyperplanes $\mathcal{H}_1, \dots, \mathcal{H}_r$, corresponding to the remaining singular points $q_1, \dots, q_r$.

Now fix a general point $t \in \mathbb{P}^*$ and let L be a general line through t. This line meets each hyperplane $\mathcal{H}_i$, $i \in \{1, \dots, r, \infty\}$, in a certain point a_i, and meets $\mathcal{Q}^*$ transversally in certain smooth points $a_{r+1}, \dots, a_s$. All the points a_i, $i \neq \infty$, correspond to irreducible hypersurfaces F_{a_i} in $\mathcal{L}$ with a unique double point. When $r+1 \leq i \leq s$, then F_{a_i} corresponds to a tangent hyperplane section of $\mathcal{Q}$, and therefore its unique double point is ordinary. The point a_∞ corresponds to a reducible hypersurface $M \cup G$ containing G:

$$F_{a_\infty} = M \cup G. \tag{20}$$

The intersection

$$B = H_t \cap H_{a_\infty} \cap \mathcal{Q} \tag{21}$$

defines the exceptional subset of $\mathcal{Q}$ with respect to L. By (19), we may regard it as a subset of $\mathcal{P}$. Denote by $\tilde{\mathcal{P}}$ the blowing-up of $\mathcal{P}$ along B, and by $\mathcal{R}$ a desingularization of $\tilde{\mathcal{P}}$. Set $f : \mathcal{R} \to L$ the natural projection. The restriction

$$f : \mathcal{R} - f^{-1}(\{a_1, \dots, a_s, a_\infty\}) \to L - \{a_1, \dots, a_s, a_\infty\} = L - \mathcal{D}$$

is a smooth proper map. Hence the fundamental group $\pi_1(L - \mathcal{D}, t)$ acts by monodromy on $f^{-1}(t) \simeq F_t$, and so on $H^{2m}(F_t; \mathbb{C})$. By the Invariant Subspace Theorem [25], p. 165-167, we know that there is an orthogonal decomposition:

$$H^{2m}(F_t; \mathbb{C}) = I \oplus V,$$

where I is the subspace of the invariant cocycles, and V is its orthogonal complement. If j denotes the natural inclusion $F_t \subset \mathcal{R}$, then we also have $I = j^* H^{2m}(\mathcal{R}; \mathbb{C})$ from which, using Poincarè duality, we get

$$V = Ker(H^{2m}(F_t; \mathbb{C}) \to H^{2m+2}(\mathcal{R}; \mathbb{C})) \simeq Ker(H_{2m}(F_t; \mathbb{C}) \to H_{2m}(\mathcal{R}; \mathbb{C})).$$

We notice that

$$Ker(H_{2m}(F_t; \mathbb{C}) \to H_{2m}(\mathcal{R}; \mathbb{C})) = Ker(H_{2m}(F_t; \mathbb{C}) \to H_{2m}(\mathcal{R} - f^{-1}(a_\infty); \mathbb{C})). \tag{22}$$

In fact, to prove (22), taking into account that $F_t \subset \mathcal{R} - f^{-1}(a_\infty)$, it suffices to prove that the natural map $H_{2m}(\mathcal{R} - f^{-1}(a_\infty); \mathbb{C}) \to H_{2m}(\mathcal{R}; \mathbb{C})$ is injective. By Lefschetz

and Poincarè dualities we have natural isomorphisms $H_{2m}(\mathcal{R} - f^{-1}(a_\infty);\mathbb{C}) \simeq H^{2m+2}(\mathcal{R}, f^{-1}(a_\infty);\mathbb{C})$ and $H_{2m}(\mathcal{R};\mathbb{C}) \simeq H^{2m+2}(\mathcal{R};\mathbb{C})$. So, to prove (22), it suffices to prove that the natural map $H^{2m+2}(\mathcal{R}, f^{-1}(a_\infty);\mathbb{C}) \to H^{2m+2}(\mathcal{R};\mathbb{C})$ is injective. This follows from the vanishing of $H^{2m+1}(f^{-1}(a_\infty);\mathbb{C})$. To prove this, using (18), first one sees that the strict transform $\tilde{G}$ of G in $\mathcal{P}$ does not meet the singular locus of $\mathcal{P}$. It follows that $f^{-1}(a_\infty)$ simply is the union $G_1 \cup M_1$ of the strict transforms of G and M in $\mathcal{R}$ (compare with (20)). Since $G_1 \simeq G$ and M_1 is isomorphic to the blowing-up of the general hypersurface $M \subset \mathbb{P}^{2m+1}$ of degree $n-k$ along the smooth complete intersection $M \cap X$ of dimension $2m-2$, and $G_1 \cap M_1 \simeq G \cap M$, then one may compute the terms of the Mayer-Vietoris sequence of the pair (G_1, M_1) which allow to control $H^{2m+1}(f^{-1}(a_\infty);\mathbb{C})$. It turns out that $H^{2m+1}(f^{-1}(a_\infty);\mathbb{C}) = 0$.

From (22) and the homology exact sequence of the pair $(\mathcal{R} - f^{-1}(a_\infty), F_t)$ we deduce

$$V \simeq Im(H_{2m+1}(\mathcal{R} - f^{-1}(a_\infty), F_t;\mathbb{C}) \to H_{2m}(F_t;\mathbb{C})). \tag{23}$$

Now, as in [18], pg. 35, Fig. 1, for any $1 \leq i \leq s$ fix a closed disk $\Delta_i \subset L - \{a_\infty\} \simeq \mathbb{C}$ with center a_i and radius $0 < \rho << 1$, and, in $\mathbb{C} - \bigcup_{i=1}^s \Delta_i^\circ$ (Δ_i° = interior of Δ_i), choose a C^∞ path l_i from t to $a_i + \rho$ with no self-intersection points and such that $l_i \cap l_j = \{t\}$ for $i \neq j$. Using the same argument as in [18], (5.3.1) and (5.3.2), one may prove a direct decomposition

$$H_{2m+1}(\mathcal{R} - f^{-1}(a_\infty), F_t;\mathbb{C}) \simeq \oplus_{i=1}^s H_{2m+1}(f^{-1}(\Delta_i), f^{-1}(a_i+\rho);\mathbb{C}).$$

If we denote by V_i the image of each $H_{2m+1}(f^{-1}(\Delta_i), f^{-1}(a_i+\rho);\mathbb{C})$ in $H_{2m}(F_t;\mathbb{C}) \simeq H^{2m}(F_t;\mathbb{C})$, then by (23) we get a decomposition:

$$V = V_1 + \cdots + V_s. \tag{24}$$

Notice that each path l_i induces a C^∞- diffeomorphism $f^{-1}(a_i+\rho) \simeq F_t$, and so an isomorphism only depending on l_i:

$$H_{2m}(f^{-1}(a_i+\rho);\mathbb{C}) \simeq H_{2m}(F_t;\mathbb{C}), \tag{25}$$

which in turn identifies

$$Im(H_{2m+1}(f^{-1}(\Delta_i), f^{-1}(a_i+\rho);\mathbb{C}) \to H_{2m}(f^{-1}(a_i+\rho);\mathbb{C})) \simeq V_i. \tag{26}$$

When $r+1 \leq i \leq s$, we recognize in $V_i \subset H^{2m}(F_t;\mathbb{C})$ the subspace generated by the "classical"vanishing cocycle corresponding to a tangent hyperplane section of $\mathcal{Q}$ (see [18], [27]). For the remaining subspaces we claim that

$$V_i = 0 \quad \text{for any}\, 1 \leq i \leq r. \tag{27}$$

To prove (27), fix an index $1 \leq i \leq r$ and denote by g the natural projection $\tilde{\mathcal{P}} \to L$, so that f is the composition of g with the desingularization $\mathcal{R} \to \tilde{\mathcal{P}}$. By [20], p.

28, we know that near to the isolated singular point $q_i \in \tilde{\mathcal{P}}$, the pencil $g : \tilde{\mathcal{P}} \to L$ defines a Milnor fibration with Milnor fiber

$$g^{-1}(a_i + \rho) \cap D_i, \tag{28}$$

where D_i denotes a closed ball of the ambient space in which $\tilde{\mathcal{P}}$ is embedded, with center q_i and positive small radius ϵ with $\rho << \epsilon << 1$. Set

$$I_i = Im(H_{2m}(g^{-1}(a_i + \rho) \cap D_i; \mathbb{C}) \to H_{2m}(g^{-1}(a_i + \rho); \mathbb{C})). \tag{29}$$

Observe that $g^{-1}(a_i + \rho)$ is canonically isomorphic to $f^{-1}(a_i + \rho)$. Hence, via l_i, by (25) we may regard V_i and I_i both contained in $H_{2m}(g^{-1}(a_i + \rho); \mathbb{C})$ and in $H_{2m}(F_t; \mathbb{C}) \simeq H^{2m}(F_t; \mathbb{C})$.

Since $g^{-1}(\Delta_i) - D_i^\circ \to \Delta_i$ is a trivial fibre bundle (D_i°= interior of D_i), using Excision Axiom and Leray-Hirsch Theorem ([26], p. 200 and 258), one sees that the inclusion $(g^{-1}(a), g^{-1}(a) \cap D_i) \subset (g^{-1}(\Delta_i), g^{-1}(\Delta_i) \cap D_i)$ induces natural isomorphisms:

$$H_{2m}(g^{-1}(a), g^{-1}(a) \cap D_i; \mathbb{C}) \simeq H_{2m}(g^{-1}(\Delta_i), g^{-1}(\Delta_i) \cap D_i; \mathbb{C}) \tag{30}$$

for any $a \in \Delta_i$. From (29), (30), the homology sequence of the pair $(g^{-1}(a_i + \rho), g^{-1}(a_i + \rho) \cap D_i)$, and the conic structure of $g^{-1}(\Delta_i) \cap D_i$ ([20], Lemma (2.10)), which implies that $H_{2m}(g^{-1}(\Delta_i), g^{-1}(\Delta_i) \cap D_i; \mathbb{C}) \simeq H_{2m}(g^{-1}(\Delta_i); \mathbb{C})$, it follows the natural exact sequence:

$$0 \to I_i \to H_{2m}(g^{-1}(a_i + \rho); \mathbb{C}) \to H_{2m}(g^{-1}(\Delta_i); \mathbb{C}). \tag{31}$$

Since we may regard the inclusion $g^{-1}(a_i + \rho) \subset g^{-1}(\Delta_i)$ as the composition of the isomorphism $g^{-1}(a_i + \rho) \simeq f^{-1}(a_i + \rho)$ with the inclusion $f^{-1}(a_i + \rho) \subset f^{-1}(\Delta_i)$, followed by the desingularization $f^{-1}(\Delta_i) \to g^{-1}(\Delta_i)$, from (26) it follows that

$$V_i \subset Ker(H_{2m}(g^{-1}(a_i + \rho); \mathbb{C}) \to H_{2m}(g^{-1}(\Delta_i); \mathbb{C}))$$

and therefore, from (31), we obtain

$$V_i \subset I_i. \tag{32}$$

Remark 3.1. Notice that from the local description (18), it follows that the singularities of $\mathcal{P}$, and hence of $\tilde{\mathcal{P}}$, are all locally complete intersection isolated singularities. Then the Milnor fiber $g^{-1}(a_i + \rho) \cap D_i$ defined by the pencil g around q_i is the Milnor fibre of the isolated complete intersection singularity $(g^{-1}(a_i) \cap D_i, q_i)$. Therefore $g^{-1}(a_i + \rho) \cap D_i$ has the homotopy type of a bouquet of $2m-$spheres contained in $g^{-1}(a_i + \rho)$, and these $2m-$spheres, as cycle classes, span I_i. In particular $H_{2m-1}(g^{-1}(a_i + \rho) \cap D_i; \mathbb{C}) = 0$, which implies that the right map in (31) actually is surjective (compare with [20], pp. 7, 73-76, 121). The number μ of $2m-$spheres occurring in the bouquet is called the Milnor number of the singularity $(g^{-1}(a_i) \cap D_i, q_i)$.

At this point we need the following basic lemma.

Lemma 3.2. *For any $1 \leq i \leq r$, the group $\pi_1(L-\mathcal{D},t)$ trivially acts on I_i.*

Proof of Lemma 3.2. Our first step consists in proving that one may assume X with a unique ordinary double point.

To this purpose, fix an integer $i \in \{1,\dots,r\}$, i.e. fix a singular point q_i of $\mathcal{Q}-\{q_\infty\}$. Consider the Hilbert scheme parametrizing all the hypersurfaces of degree n in $\mathbb{P}^5$:

$$\mathcal{F} \subset \mathbb{P}^{2m+1} \times \mid \mathcal{O}_{\mathbb{P}^{2m+1}}(n) \mid \to \mid \mathcal{O}_{\mathbb{P}^{2m+1}}(n) \mid . \tag{33}$$

Notice that by the rational map defined in (17), we may regard L as a line in $\mid \mathcal{O}_{\mathbb{P}^{2m+1}}(n) \mid$ and the restriction of the universal family (33) to L gives our pencil $g : \tilde{\mathcal{P}} \to L$, i.e. we have:

$$\tilde{\mathcal{P}} = \mathcal{F} \times_{|\mathcal{O}_{\mathbb{P}^{2m+1}}(n)|} L \subset \mathbb{P}^{2m+1} \times \mid \mathcal{O}_{\mathbb{P}^{2m+1}}(n) \mid . \tag{34}$$

It follows that the Milnor fibre defined by g in correspondence of the critical value $a_i \in L$ is equal to the Milnor fibre defined by the universal family (33) at a_i. Therefore, we may interpret the Milnor number μ of the singularity of the hypersurface F_{a_i} (recall Remark (3.1)) as the multiplicity of the discriminant locus $\mathcal{D}_n$ of the whole linear system $\mid \mathcal{O}_{\mathbb{P}^{2m+1}}(n) \mid$ at a_i (see [20], pp. 63-64 and 77, and [10], p. 81). Hence, a general line $L' \subset \mid \mathcal{O}_{\mathbb{P}^{2m+1}}(n) \mid$ passing near to the critical value a_i of L ($a_i \notin L'$) transversally meets $\mathcal{D}_n$ in μ smooth points $b_1,\dots,b_\mu$, and the singular locus of each hypersurface F_{b_h} consists of exactly one node. Similarly as in the definition of I_i (see (29)), using this node we may define a certain cocycle δ_{b_h} in $H^{2m}(F_t;\mathbb{C})$. It turns out that these cocycles $\{\delta_{b_1},\dots,\delta_{b_\mu}\}$ lie in I_i and here they form a distinguished basis (see [20], p. 76, and [10], p. 83). And so to prove our Lemma 3.2 it suffices to prove that the monodromy induced by L trivially acts on each δ_{b_h}.

To this purpose, fix a δ_{b_h}, and choose general germs $\{F_\tau\}_{\tau\in\Delta}$ and $\{G_\tau\}_{\tau\in\Delta}$ in $\mid \mathcal{O}_{\mathbb{P}^{2m+1}}(n) \mid$ and in $\mid \mathcal{O}_{\mathbb{P}^{2m+1}}(k) \mid$, with $F_0 = F_t$, $F_\epsilon = F_{b_h}$, $G_0 = G$, such that, for any $\tau \neq 0$, F_τ only has one node belonging also to G_τ (here $\Delta \subset \mathbb{C}$ denotes a closed disk centered at 0 with small radius ϵ). Put $X_\tau = G_\tau \cap F_\tau$. Then X_τ is a complete intersection of dimension $2m-1$ with only one node, and, as for $X = X_0$, in correspondence of each X_τ we may define a general pencil $g_\tau : \tilde{\mathcal{P}}_\tau \to L_\tau$, with $L_\tau \subset \mid \mathcal{I}_{X_\tau,\mathbb{P}^{2m+1}}(n) \mid \subset \mid \mathcal{O}_{\mathbb{P}^{2m+1}}(n) \mid$, in such a way that the family $\{L_\tau\}_{\tau\in\Delta}$ trivially deforms our starting pencil $L = L_0$. Similarly as in the definition of I_i, the Milnor fibre of g_ϵ corresponding to the nodal fibre $F_\epsilon = F_{b_h} = g_\epsilon^{-1}(b_h)$, defines a subspace $I_\epsilon \subset H_{2m}(g_\epsilon^{-1}(b_h+\rho_\epsilon);\mathbb{C})$ ($0 < \rho_\epsilon << 1$). Notice that, via the total space of the family $\{L_\tau\}_{\tau\in\Delta}$, we may transport I_ϵ in $H_{2m}(F_t;\mathbb{C}) \simeq H^{2m}(F_t;\mathbb{C})$, and we may assume that in this way we obtain exactly $Span(\delta_{b_h})$. Moreover, using

again the total space of the deformation $\{L_\tau\}_{\tau\in\Delta}$, we see that any closed path in $L-\mathcal{D}$ is free homotopic in $\mid \mathcal{O}_{\mathbb{P}^{2m+1}}(n) \mid -\mathcal{D}_n$ to some closed path contained in $L_\epsilon - \mathcal{D}_\epsilon$ ($\mathcal{D}_\epsilon$ = discriminant locus of $\mid \mathcal{I}_{X_\tau,\mathbb{P}^{2m+1}}(n) \mid$). It follows that if the monodromy of L_ϵ trivially acts on I_ϵ, then also the monodromy of L trivially acts on δ_{b_h}. Therefore, in order to prove Lemma 3.2, we may assume X with a unique ordinary double point.

With this assumption, then $Sing(\mathcal{Q}) = \{q_1, q_\infty\}$, and we only have to prove that the monodromy defined by L trivially acts on I_1. To this purpose, let $\pi \subset \mathbb{P}^*$ be a general projective plane, so that $\pi \cap \mathcal{H}_1$ is a general line in $\mathcal{H}_1$. Denote by Y the set of points in $\pi\cap\mathcal{H}_1$ parametrizing hyperplanes which are limit of some sequence z_n of tangent hyperplanes at smooth part of $\mathcal{Q}$, such that there exists a sequence of regular contact points $p_n \in Sing(\mathcal{Q}\cap H_{z_n})$ converging to q_1. Notice that Y is contained in the finite set $\pi\cap\mathcal{H}_1\cap\mathcal{Q}^*$. For any $y \in Y$ denote by Δ_y a closed disk of $\pi\cap\mathcal{H}_1$, with center y and positive radius $<< 1$, and put $K = (\pi\cap\mathcal{H}_1) - \bigcup_{y\in Y} \Delta_y^\circ$. Notice also that we may assume our pencil L contained in π and close to $\pi\cap\mathcal{H}_1$, because such a pencil is sufficiently general to apply Zariski Theorem, which ensures that $\pi_1(L-\mathcal{D}, t)$ maps onto $\pi_1(\mathbb{P}^*-\mathcal{D}, t)$ (see [18], (7.4.1), or [27], Théorème 15.22). Now consider the restriction of the universal family (33) to π:

$$\varphi : \mathcal{F}_\pi := \mathcal{F} \times_{|\mathcal{O}_{\mathbb{P}^{2m+1}}(n)|} \pi \to \pi.$$

Recall from (34) that we may regard $\tilde{\mathcal{P}} \subset \mathcal{F}_\pi$, and the pencil $g : \tilde{\mathcal{P}} \to L$ as the restriction of φ to L. Using [20], Theorem (2.8), we see that for any $x \in K$ there exists a closed ball $D_{q_1,x} \subset \mathbb{P}^{2m+1} \times \mid \mathcal{O}_{\mathbb{P}^{2m+1}}(n) \mid$, with positive radius and centered at q_1, and a closed ball $C_x \subset \pi$ (with positive radius and of real dimension 4) centered at x, such that the induced map

$$\varphi_x : \varphi^{-1}(C_x) \cap D_{q_1,x} \to C_x$$

is a Milnor fibration whose discriminant locus simply is $C_x \cap \mathcal{H}_1$ (observe that $\pi \cap \mathcal{H}_1 \cap (\mathcal{Q}^* \cup \mathcal{H}_\infty) = Y \cup \{pt\}$, with $\mid Y \mid= 2$ and $pt \notin Y$). In view of (34), if the critical value a_1 of L corresponding to q_1 belongs to some C_x, then the Milnor fibre as defined in (28) (note that now we have $i = 1$) is the Milnor fibre of φ_x.

Moreover, since $x \in K$ then we may assume that for any $z \in C_x \cap \mathcal{H}_1$ the map φ_x represents the Milnor fibration of the isolated complete intersection singularity $(\varphi^{-1}(z)\cap D_{q_1,x}, q_1)$. Therefore, since K is compact, using the local data $x \in K$, C_x and $D_{q_1,x}$, one may construct a connected open tubular neighborhood $\mathcal{M}$ of K in π, with $a_1 \in \mathcal{M}$, and a closed ball $D_{q_1} \subset \mathbb{P}^{2m+1} \times \mid \mathcal{O}_{\mathbb{P}^{2m+1}}(n) \mid$ of positive radius and centered at q_1 such that the map

$$\varphi_{\mathcal{M}} : z \in \varphi^{-1}(\mathcal{M}) \cap D_{q_1} \to \varphi(z) \in \mathcal{M} \tag{35}$$

defines a C^∞-fibre bundle on $\mathcal{M} - \mathcal{H}_1$, and whose fibre $\varphi_{\mathcal{M}}^{-1}(z)$, $z \in \mathcal{M} - \mathcal{H}_1$, may be identified with the Milnor fibre of $g : \tilde{\mathcal{P}} \to L$ corresponding to q_1.

Since also $\pi \cap \mathcal{H}_1$ is compact, one may construct an open tubular neighborhood $\mathcal{N}$ of $\pi \cap \mathcal{H}_1$ in π in such a way that one may obtain $\mathcal{M}$ removing from $\mathcal{N}$ suitable compact tubular neighborhoods $\mathcal{N}_y$ of the disks Δ_y, $y \in Y$.

Now, as in [18], p. 35, Fig. 1, for any critical value a_j of L fix a closed disk $\Delta_j \subset L - \{t\} \simeq \mathbb{C}$ with center a_j and radius $0 < \rho << 1$, and, in $L - \bigcup_j \Delta_j^\circ$ ($\Delta_j^\circ =$ interior of Δ_j), choose a C^∞ path l_j from t to $a_j + \rho$ with no self-intersection points and such that $l_j \cap l_h = \{t\}$ for $j \neq h$. Let $w_j \in \pi_1(L - \mathcal{D}, t)$ be the homotopy class defined by the path $l_j^{-1} \cdot \partial\Delta_j \cdot l_j$. These classes span $\pi_1(L - \mathcal{D}, t)$ with exactly one relation

$$w_1 \cdot w_2 \cdots w_s \cdot w_\infty = 1.$$

Therefore, to prove Lemma 3.2 it suffices to prove that any w_j, $j \neq 1$, trivially acts on I_1.

From the definition of $\mathcal{N}$ we may assume that all critical values of L are contained in $\mathcal{N}$. Now, with the exception of the point $\{a_1\} = L \cap \mathcal{H}_1 \cap \mathcal{M}$, in a neighborhood of any critical value a_j of L lying in $\mathcal{M}$, the fibration $\varphi_{\mathcal{M}}$ is trivial. Since $g : \tilde{\mathcal{P}} \to L$ is a subbundle of φ, and the fibre $\varphi_{\mathcal{M}}^{-1}(z)$, $z \in \mathcal{M} - \mathcal{H}_1$, identifies with the Milnor fibre of g corresponding to q_1, then w_j trivially acts on I_1. Therefore, to complete the proof of Lemma 3.2, it remains to analyze the monodromy corresponding to the set $\mathcal{D}(L, y)$ of the critical values of L contained in each $\mathcal{N}_y$.

To this aim, denote by Σ the set of points in $\mathcal{H}_1$ which are limit of some sequence z_n of tangent hyperplanes at smooth part of $\mathcal{Q}$, such that there exists a sequence of regular contact points $p_n \in Sing(\mathcal{Q} \cap H_{z_n})$ converging to q_1. Since q_1 is an ordinary double point for $\mathcal{Q}$, then Σ is the dual variety of the tangent cone of $\mathcal{Q}$ at q_1. Hence Σ is a nondegenerate irreducible quadric in $\mathcal{H}_1$. Note that $Y = \pi \cap \Sigma$, and $|Y| = 2$. Now recall that X is a complete intersection of a smooth hypersurface in $\mathbb{P}^{2m+1}$ with a general hypersurface with a unique ordinary double point. Hence, for a general point $y \in \Sigma$, $H_y \cap \mathcal{Q}$ has an isolated singular point at q_1 with Milnor number 2. Combining ([20], pp. 62-63) with ([20], (5.11.a), p. 77), we know that for a general line $L' \subset \mathbb{P}^*$ passing through such a point y, the intersection multiplicity $m_y(L', \mathcal{D})$ of L' with $\mathcal{D}$ at y is the sum of the Milnor number of $H_y \cap \mathcal{Q}$ at q_1, with the Milnor number of $\mathcal{Q}$ at q_1. Therefore we have $m_y(L', \mathcal{D}) = 3$. On the other hand, the same argument applied to the general point x of $\mathcal{H}_1$, which parametrizes a hyperplane section with an ordinary double point at q_1, shows that for the general line $L'' \subset \mathbb{P}^*$ passing through x one has $m_x(L'', \mathcal{D}) = 2$. It follows that the critical locus of our pencil L meets $\mathcal{N}_y$ in exactly one point, corresponding to some tangent hyperplane to the smooth part of $\mathcal{Q}$.

Since for any $y \in Y$ we have $|\mathcal{D}(L, y)| = 1$, then our argument above involving the fibration (35) proves that I_1 is at least globally invariant under the monodromy action induced by L. Now denote by $\mathcal{T}(L)$ the set of critical values of L corresponding to tangent hyperplane sections of $\mathcal{Q}$, and fix a point $a_{j_0} \in \mathcal{T}(L) \cap \mathcal{M}$. For

such a critical value a_{j_0} we just proved that the homotopy class w_{j_0} trivially acts on I_1. Therefore, if we denote by $\delta_{j_0} \in H^{2m}(F_t;\mathbb{C})$ the "classical" vanishing cocycle generating V_{j_0}, from Picard-Lefschetz formula it follows that for any $\xi \in I_1$ one has $<\xi, \delta_{j_0}>= 0$. On the other hand, from ([27], Proposition 15.23) and ([20], p. 113, Lemma (7.2)) we know that $\pi_1(L - \mathcal{Q}^*, t)$ irreducibly acts on the cocycles determined by tangent hyperplane sections of $\mathcal{Q}$. A fortiori this holds true for $\pi_1(L - \mathcal{D}, t)$ and so, from the global invariance of I_1, we deduce that for any $\xi \in I_1$ and any $a_j \in \mathcal{T}(L)$ one has $<\xi, \delta_j>= 0$. From the Picard-Lefschetz formula again it follows that for any $a_j \in \mathcal{T}(L)$ the homotopy class w_j trivially acts on I_1. Since we proved that for any $y \in Y$ one has $\mathcal{D}(L, y) \subset \mathcal{T}(L)$, this concludes the proof of Lemma 3.2. □

From Lemma 3.2, (24) and (32) we get $V_i \subset I \cap V = 0$, and this proves (27). In other words we have:

$$V = V_{r+1} + \cdots + V_s.$$

This means that V is generated by the vanishing cocycles determined by the hyperplanes of the pencil L which are tangent to the smooth part of Q. Therefore, as before (see ([27], Proposition 15.23) and ([20], p. 113, Lemma (7.2))) $\pi_1(L - \mathcal{D}, t)$ irreducibly acts on V.

Remark 3.3. This concludes the proof of Theorem 1.5.

This enables us to prove that:

$$NS_m(F_t) \subset I \tag{36}$$

(as before, we identify $H_{2m}(F_t;\mathbb{C}) \simeq H^{2m}(F_t;\mathbb{C})$ via Poincarè duality). In fact, argue by contradiction. Suppose there exists $\xi \in NS_m(F_t)$ such that $\xi^w \neq \xi$ for some $w \in \pi_1(L - \mathcal{D}, t)$. We may write $\xi = i + v$ for some $i \in I$ and $v \in V$, and we have

$$v - v^w = \xi - \xi^w \neq 0. \tag{37}$$

Since $\pi_1(L - \mathcal{D}, t)$ irreducibly acts on V, and $NS_m(F_t)$ is globally invariant, (37) implies that $V \subset H^{m,m}(F_t, \mathbb{C})$. On the other hand $\mathcal{R}$ is birational to $\mathbb{P}^{2m+1}$ and so $H^{2m,0}(\mathcal{R}, \mathbb{C}) = 0$ (see [15], p. 190, Ex. 8.8). Therefore, since $I = j^*H^4(\mathcal{R};\mathbb{C})$, we get

$$H^{2m,0}(F_t;\mathbb{C}) = I^{2m,0} \oplus V^{2m,0} = 0.$$

This is in contrast with our hypothesis $n > 2m + 1$. This proves (36).

We are in position to prove that the natural map $NS_m(X) \to NS_m(F_t)$ is surjective. To this purpose fix an algebraic class $\xi \in NS_m(F_t)$, which we may assume represented by some projective algebraic subvariety $S_1 \subset F_t$ of dimension m, and consider the flag Hilbert scheme $\mathcal{S}$, with reduced structure, parametrizing

pairs (S, F), with $F \in \mathcal{L}$ and $S \subset F$ a projective subvariety of dimension m. Let $\mathcal{C} \subset \mathcal{S}$ be an irreducible projective curve passing through the point (S_1, F_t). Since F_t is Noether-Lefschetz general, we may assume $\mathcal{C}$ dominating L and such that t is a regular value of the natural branched covering map $\pi : \mathcal{C} \to L$. This curve determines a projective subvariety $T \subset \tilde{\mathcal{P}}$ of dimension $m+1$, whose intersection with F_t is the union of all the subvarieties S_i, $i = 1, \dots, d$, corresponding to the fibre of π over the point $t \in L$ ($d =$ degree of π). The monodromy of π is transitive, and so by (36) we deduce that all the S_i are homologous in F_t. In other words, we have

$$\xi = \frac{1}{d} \cdot j_1^*(T) \quad \text{in} \quad NS_m(F_t), \tag{38}$$

where $j_1^* : A_{m+1}(\tilde{\mathcal{P}}) \to NS_m(F_t)$ is the natural map induced by the inclusion $F_t \subset \tilde{\mathcal{P}}$. Now recall from (21) that $\tilde{\mathcal{P}}$ is the blowing-up of $\mathcal{P}$ along the base locus B, which is isomorphic to a projective smooth complete intersection in $\mathbb{P}^{2m+1}$ of dimension $2m-1$. By ([12], p.114-115, Proposition 6.7, (e)), we know that $A_{m+1}(B \times \mathbb{P}^1) \oplus A_{m+1}(\mathcal{P})$ maps onto $A_{m+1}(\tilde{\mathcal{P}})$, and therefore we may write

$$T = lH^m_{\tilde{\mathcal{P}}} + \alpha^*(Z) \quad \text{in} \quad A_{m+1}(\tilde{\mathcal{P}}),$$

where $H_{\tilde{\mathcal{P}}}$ is the pull-back in $\tilde{\mathcal{P}}$ of the hyperplane class in $\mathbb{P}^{2m+1}$, l is a suitable integer, $\alpha : \tilde{\mathcal{P}} \to \mathcal{P}$ is the natural projection, and Z is a suitable class in $A_{m+1}(\mathcal{P})$. Plugging previous formula into (38), and using the natural map $j_2^* : A_{m+1}(\mathcal{P}) \to NS_m(F_t)$ induced by the inclusion $F_t \subset \mathcal{P}$, we get

$$j_1^*(T) = j_2^*(lH^m_{\mathcal{P}} + Z), \tag{39}$$

where $H_{\mathcal{P}}$ is the pull-back in $\mathcal{P}$ of the hyperplane class in $\mathbb{P}^{2m+1}$. Since $\mathcal{P}$ is the blowing-up of $\mathbb{P}^{2m+1}$ along X, again by ([12], l.c.) we know that $A_{m+1}(\tilde{X}) \oplus A_{m+1}(\mathbb{P}^{2m+1})$ maps onto $A_{m+1}(\mathcal{P})$, where $h : \tilde{X} \subset \mathcal{P}$ is the exceptional divisor, which in turn is a $\mathbb{P}^1-$bundle $\beta : \tilde{X} \to X$ over the complete intersection X. The group $A_{m+1}(\tilde{X})$ is spanned by $\beta^*(A_m(X))$, and by the cycles obtained intersecting a fixed section of β with $\beta^*(A_{m+1}(X))$. As a section we may choose $\tilde{G} \cap \tilde{X}$ (compare with (19)). It follows that we may write

$$lH^m_{\mathcal{P}} + Z = lH^m_{\mathcal{P}} + h_*\beta^*(W_1) + h_*(\tilde{G} \cap \tilde{X} \cap \beta^*(W_2)) \quad \text{in} \quad A_{m+1}(\mathcal{P}),$$

where W_1 and W_2 are suitable classes in $A_m(X)$ and $A_{m+1}(X)$. Taking into account that $\tilde{G} \cap \tilde{X}$ is disjoint with F_t, from (38) and (39) we obtain

$$\xi = \frac{1}{d} \cdot j_2^*(lH^m_{\mathcal{P}} + h_*\beta^*(W_1) + h_*(\tilde{G} \cap \tilde{X} \cap \beta^*(W_2))) = \frac{1}{d} \cdot \gamma_*(\frac{l}{k} H_X^{m-1} + W_1),$$

where H_X is the hyperplane section of X, $k = deg(G)$ and γ_* denotes the map $NS_m(X) \to NS_m(F_t)$. This proves that this map is onto, and concludes the proof of Theorem 1.2. $\square$

Proof of Theorem 1.8. We are in position to prove Theorem 1.8 stated in the Introduction. To this purpose consider a general pencil of hypersurface sections $\{\mathcal{Q} \cap R_t\}_{t \in \mathbb{P}^1}$ of $\mathcal{Q}$, with $deg(R_t) >> 0$. With the same methods we used in the proof of Theorem 1.5, for a general t, we may prove an orthogonal decomposition $H^{2m}(\mathcal{Q} \cap R_t; \mathbb{C}) = I \oplus V$, such that the monodromy representation of the pencil is irreducible on V, and $I = j^* H^{2m}(\mathcal{R}; \mathbb{C})$, where $\mathcal{R}$ denotes a certain desingularization of $\mathcal{Q}$ and j the inclusion $\mathcal{Q} \cap R_t \subset \mathcal{R}$. Since $deg(R_t) >> 0$ then $h^{2m,0}(\mathcal{Q} \cap R_t) > h^{2m,0}(\mathcal{R})$ (racall that $h^{2m,0}(\mathcal{R})$ is a birational invariant, and so it only depends on $\mathcal{Q}$). It follows that $NS_m(\mathcal{Q} \cap R_t) \subset I$. A similar argument as in the proof of (38) shows that $NS_{m+1}(\mathcal{R})$ maps onto $NS_m(\mathcal{Q} \cap R_t)$. On the other hand, since R_t does not meet the singular locus of $\mathcal{Q}$, then for the Gysin morphisms $a : H_{2m+2}(\mathcal{R}; \mathbb{C}) \to H_{2m}(\mathcal{Q} \cap R_t; \mathbb{C})$ and $b : H_{2m+2}(\mathcal{Q}; \mathbb{C}) \to H_{2m}(\mathcal{Q} \cap R_t; \mathbb{C})$ one has $a = b \circ p$, where $p : H_{2m+2}(\mathcal{R}; \mathbb{C}) \to H_{2m+2}(\mathcal{Q}; \mathbb{C})$ denotes the push-forward. Therefore also $NS_{m+1}(\mathcal{Q})$ maps onto $NS_m(\mathcal{Q} \cap R_t)$, and so $dim(NS_m(\mathcal{Q} \cap R_t)) = 1$. This concludes the proof of Theorem 1.8. □

Proof of Theorem 1.7. With the same notation as in the proof of Theorem 1.2, we are going to prove that the image I_X of $H_{2m}(X; \mathbb{C})$ in $H_{2m}(F; \mathbb{C}) \simeq H^{2m}(F; \mathbb{C})$ is equal to I. First notice that $I_X \subset I$ because the cycles coming from X are invariant. So it suffices to prove that $I \subset I_X$. Since $I = j^* H^{2m}(\mathcal{R}; \mathbb{C})$, via Poincarè duality we see that I is equal to the image of the Gysin morphism $a : H_{2m+2}(\mathcal{R}; \mathbb{C}) \to H_{2m}(F; \mathbb{C})$. Since F, as subvariety of $\mathcal{P}$, does not meet the singular locus of $\mathcal{P}$, then one has $a = b \circ p$, where $p : H_{2m+2}(\mathcal{R}; \mathbb{C}) \to H_{2m+2}(\mathcal{P}; \mathbb{C})$ denotes the push-forward and b the Gysin morphism $H_{2m+2}(\mathcal{P}; \mathbb{C}) \to H_{2m}(F; \mathbb{C})$. Therefore I is contained in the image of $H_{2m+2}(\mathcal{P}; \mathbb{C})$ through b. Now denote by $\tilde{X}$ the exceptional divisor of $\mathcal{P}$. From [18], p. 23, we know there exists a natural isomorphism $H_*(\mathcal{P}, \tilde{X}; \mathbb{C}) \simeq H_*(\mathbb{P}^{2m+1}, X; \mathbb{C})$. On the other hand, using [10], Theorem 4.3, p. 161, one sees that $H_{2m+2}(\mathbb{P}^{2m+1}, X; \mathbb{C}) = H_{2m+3}(\mathbb{P}^{2m+1}, X; \mathbb{C}) = 0$. Hence the inclusion $\tilde{X} \subset \mathcal{P}$ induces a natural isomorphism $H_{2m+2}(\mathcal{P}; \mathbb{C}) \simeq H_{2m+2}(\tilde{X}; \mathbb{C})$, and so $H_{2m+2}(\tilde{X}; \mathbb{C})$ maps onto I. Taking into account that $\tilde{X}$ is a $\mathbb{P}^1$-bundle over X, from Leray-Hirsch Theorem ([26], p. 258) we know that all the homology of $\tilde{X}$ comes from X, up the cycles contained in a fixed section of the bundle $\tilde{X} \to X$, which we may choose disjoint with F. Therefore I_X contains I, and this concludes the proof of Theorem 1.7. □

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